AIMS TEACHER TRAINING PROGRAM (TTP) IN PARTNERSHIP WITH MASTERCARD FOUNDATION AND THE GOVERNMENT OF CAMEROON

MATHEMATICS OLYMPIAD

LEVEL: NATIONAL

DATE: 15TH MAY 2021

DURATION OF PAPER: 2 hours

CANDIDATES: UPPER SIXTH

PART B

INSTRUCTIONS TO CANDIDATES:

- Mobile phones are **NOT ALLOWED** in the examination room
- You should attempt to answer all questions.
- You are reminded of the necessity for orderly presentation and good English in your work.
- In calculations, you are advised to show all steps in your work, and show answers at each stage
- Non-programmable electronic calculators are allowed
- Graph paper will be provided

INSTRUCTIONS: ANSWER ALL FOUR QUESTIONS IN THIS SECTION. EACH QUESTION CARRIES 15 MARKS

1 (i) Find the number of solutions in the set of positive integers of the following equations:

a) $x + y = 14$

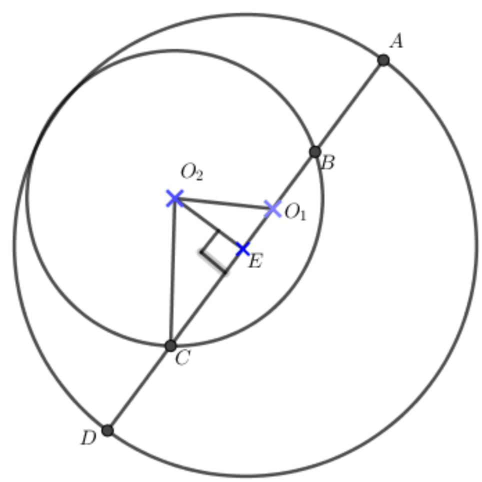
b) $x + y + z = 5$

(ii) Solve simultaneously, the linear congruences:

$$\begin{cases} x \equiv 1(mod3) \\ x = 5(mod8) \\ x = 11(mod17) \end{cases}$$

(iii) Show that the equation $x^2 - y^2 = 74$ has no integer solution

2 Two circles are internally tangent. A line passing through the center(O_1) of the larger circle intersects it at the points A and D.The same line intersects the smaller circle with center(O_2) at the points B and C as shown below. Given that $|AB|:|BC|:|CD| = 3:7:2$, find the ratio of the radii of the circles.



3 The Figure beside shows the curve of

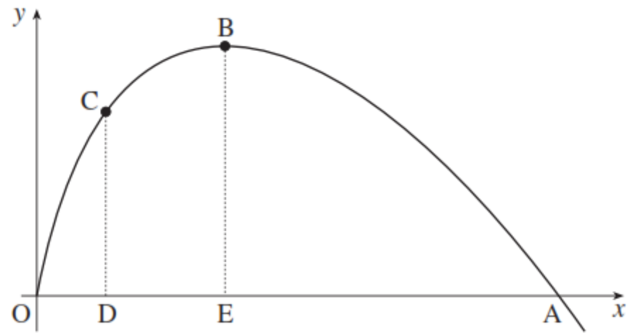
$\boldsymbol{y = 2x - x\ln x \text{, where } x > 0}$

The curve crosses the *x*-axis at A, and has a turning point at B. The point C on the curve has *x*-coordinate 1. Lines CD and BE are parallel to the *y*-axis.

i) Find the *x*-coordinate of A, giving your answer in terms of e.
ii) Find the exact coordinates of B.
iii) Show that the tangents at A and C are perpendicular to each other.
iv) Using integration by parts, show that

$$\int x\ln x dx = \frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C$$

Hence find the exact area of the region enclosed by the curve, the *x*-axis and the lines CD and BE



4 i)Given that $I_n = \int_0^{\frac{\pi}{2}} e^{-x}\cos^n x dx$, where $n \geq 2$, prove that:

a) $I_n = 1 - n\int_0^{\frac{\pi}{2}} e^{-x}\sin x\cos^{n-1} x dx$
b) $(n^2+1)I_n = 1 + n(n-1)I_{n-2}$
c) $I_6 = \frac{263 - 144e^{-\frac{\pi}{2}}}{629}$

ii) Test whether the series $\sum_{n=0}^{\infty}\left(\frac{2^{n-1}}{4+n}\right)$ converges or not.